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M.G.V.C ARTS, COMMERCE AND SCIENCE COLLEGE MUDDEBIHAL

DEPARTMENT OF MATHEMATICS

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Topic- NEWTON RAPHSON METHOD

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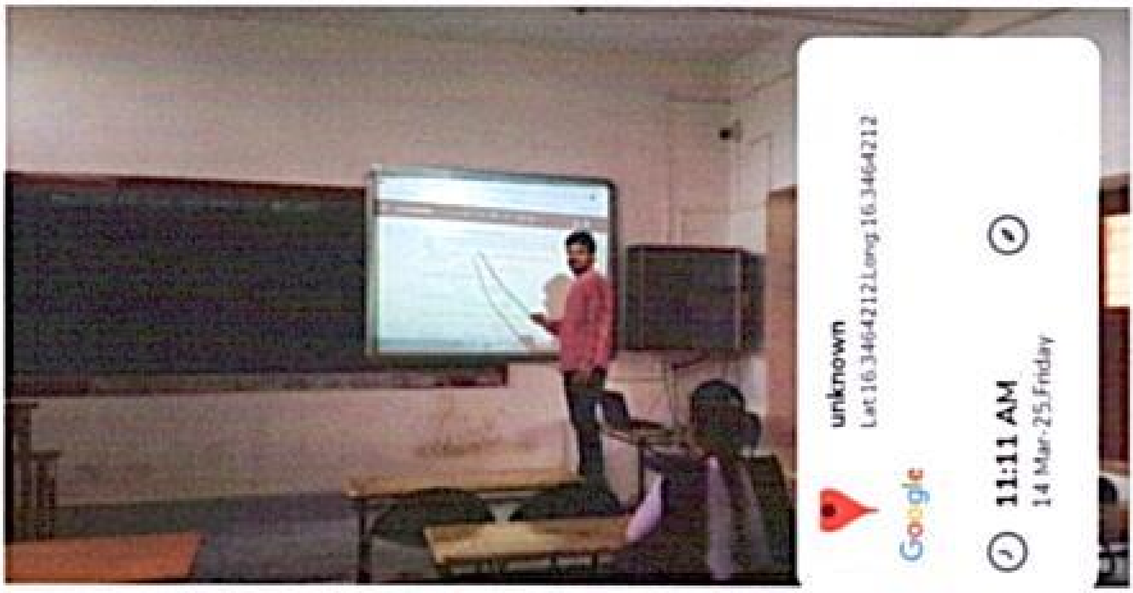

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The Newton-Raphson Method

1 Introduction

The Newton-Raphson method, or Newton Method, is a powerful technique for solving equations numerically. Like so much of the differential calculus, it is based on the simple idea of linear approximation. The Newton Method, properly used, usually homes in on a root with devastating efficiency.

The essential part of these notes is Section 2.1, where the basic formula is derived, Section 2.2, where the procedure is interpreted geometrically, and—of course—Section 6, where the problems are. Peripheral but perhaps interesting is Section 3, where the birth of the Newton Method is described.

2 Using Linear Approximations to Solve Equations

Let $f(x)$ be a well-behaved function, and let r be a root of the equation $f(x) = 0$. We start with an estimate x_0 of r. From x_0, we produce an improved—we hope—estimate x_1. From x_1, we produce a new estimate x_2. From x_2, we produce a new estimate x_3. We go on until we are 'close enough' to r—or until it becomes clear that we are getting nowhere.

The above general style of proceeding is called *iterative*. Of the many iterative root-finding procedures, the Newton-Raphson method, with its combination of simplicity and power, is the most widely used. Section 2.4 describes another iterative root-finding procedure, the *Secant Method*.

Comment. The initial estimate is sometimes called x_1, but most mathematicians prefer to start counting at 0.

Sometimes the initial estimate is called a "guess." The Newton Method is usually very very good if x_0 is close to r, and can be horrid if it is not. The "guess" x_0 should be chosen with care.

2.1 The Newton-Raphson Iteration

Let x_0 be a good estimate of r and let $r = x_0 + h$. Since the true root is r, and $h = r - x_0$, the number h measures how far the estimate x_0 is from the truth.

Since h is 'small,' we can use the linear (tangent line) approximation to conclude that

$$0 = f(r) = f(x_0 + h) \approx f(x_0) + hf'(x_0),$$

and therefore, unless $f'(x_0)$ is close to 0,

$$h \approx -\frac{f(x_0)}{f'(x_0)}.$$

It follows that

$$r = x_0 + h \approx x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Our new improved (?) estimate x_1 of r is therefore given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The next estimate x_2 is obtained from x_1 in exactly the same way as x_1 was obtained from x_0:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Continue in this way. If x_n is the current estimate, then the next estimate x_{n+1} is given by

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad (1)$$

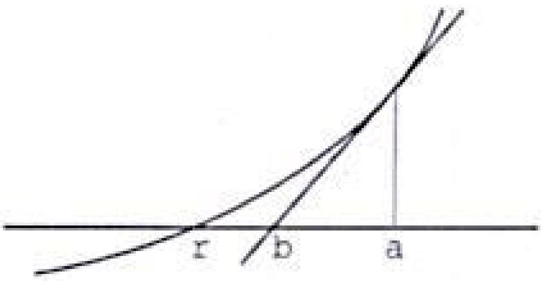
2.2 A Geometric Interpretation of the Newton-Raphson Iteration

In the picture below, the curve $y = f(x)$ meets the x-axis at r. Let a be the current estimate of r. The tangent line to $y = f(x)$ at the point $(a, f(a))$ has equation

$$y = f(a) + (x - a)f'(a).$$

Let b be the x-intercept of the tangent line. Then

$$b = a - \frac{f(a)}{f'(a)}.$$



Compare with Equation 1: b is just the 'next' Newton-Raphson estimate of r. The new estimate b is obtained by drawing the tangent line at $x = a$, and then sliding to the x-axis along this tangent line. Now draw the tangent line at $(b, f(b))$ and ride the new tangent line to the x-axis to get a new estimate c. Repeat.

We can use the geometric interpretation to design functions and starting points for which the Newton Method runs into trouble. For example, by putting a little bump on the curve at $x = a$ we can make b fly far away from r. When a Newton Method calculation is going badly, a picture can help us diagnose the problem and fix it.

It would be wrong to think of the Newton Method simply in terms of tangent lines. The Newton Method is used to find complex roots of polynomials, and roots of systems of equations in several variables, where the geometry is far less clear, but linear approximation still makes sense.

2.3 The Convergence of the Newton Method

The argument that led to Equation 1 used the informal and imprecise symbol $\approx$. We probe this argument for weaknesses.

No numerical procedure works for *all* equations. For example, let $f(x) = x^2 + 17$ if $x \neq 1$, and let $f(1) = 0$. The behaviour of $f(x)$ *near* 1 gives no clue to the fact that $f(1) = 0$. Thus no method of successive approximation can arrive at the solution of $f(x) = 0$. To make progress in the analysis, we need to assume that $f(x)$ is in some sense smooth. We will suppose that $f''(x)$ (exists and) is continuous near r.

The tangent line approximation is—*an approximation*. Let's try to get a handle on the error. Imagine a particle travelling in a straight line, and let $f(x)$ be its position at time x. Then $f'(x)$ is the velocity at time x. If the acceleration of the particle were always 0, then the *change* in position from time x_0 to time $x_0 + h$ would be $hf'(x_0)$. So the position at time $x_0 + h$